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FABER-HYPERCYCLIC OPERATORS

BY

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ABSTRACT

Let Ω be a bounded domain of the complex plane whose boundary is a closed Jordan curve and $(F_n)_{n\geq 0}$ the sequence of Faber polynomials of Ω. We say that a bounded linear operator T on a separable Banach space X is Ω-hypercyclic if there exists a vector x of X such that $\{F_n(T)x : n \geq 0\}$ is dense in X. We show that many of the results in the spectral theory of hypercyclic operators involving the unit disk or its boundary have Ω-hypercyclic counterparts which involve the domain Ω or its boundary. The influence of the geometry of Ω or the smoothness of its boundary on Faber-hypercyclicity is also discussed.

1. Introduction

1.1. Motivation. Let X be a complex infinite dimensional separable Banach space and $T \in \mathcal{B}(X)$ a bounded linear operator on X. Our aim in this paper is to study a modification of the classical notion of hypercyclicity. The operator T is said to be **hypercyclic** if there exists a vector $x \in X$ such that $\{T^n x : n \geq 0\}$ is dense in X. This property of the dynamical system (X, T) has recently been a subject of an extensive study , and we refer the reader to two surveys [15] and [16] for a description of the main results in this area.

The motivation of our study comes from the following observation: there are several results in the spectral theory of hypercyclic operators involving the unit disk $\mathbb{D}$ or the unit circle $\mathbb{T}$. A possible explanation for this frequent occurrence

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is given by the following remark. The unit disk is hidden in the definition of a hypercyclic operator in the sense that the iterates T^n coincide with $F_n^{\mathbb{D}}(T)$, where $F_n^{\mathbb{D}}(z) = z^n$ represent the basic Taylor polynomials associated to $\mathbb{D}$. Let now Ω be a non-empty connected open subset of $\mathbb{C}$ with compact closure $\overline{\Omega}$ and rectifiable boundary $\partial\Omega$. The Faber polynomials F_n^{Ω} associated to the domain Ω are a natural generalization of the Taylor polynomials of the disk, and they are fundamental in many questions of complex approximation (see for instance [25]). Among the simplest results of the theory is the fact that any function which is analytic in a neighborhood of $\overline{\Omega}$ can be expanded as a series $\sum_{n\geq 0} a_n F_n^{\Omega}$ for appropriate complex scalars a_n, which is uniformly convergent on $\overline{\Omega}$ (see for instance [25, p. 52]). For the precise definition and a survey of the main properties of the Faber polynomials we refer the reader to Section 2 below. The following definition is now natural:

Definition 1.1: A bounded operator T on X is said to be **Ω-hypercyclic** if there exists a vector x of X such that $\{F_n^{\Omega}(T)x : n \geq 0\}$ is dense in X. Such a vector is an Ω-hypercyclic vector for T, and the set of such vectors is denoted by $HC_{\Omega}(T)$.

Then $\mathbb{D}$-hypercyclicity is exactly the same notion as hypercyclicity. Saying that T is Ω-hypercyclic is equivalent to requiring that the sequence $(F_n^{\Omega}(T))_{n\geq 0}$ be universal (see [15]). Thus several properties of universal sequences apply to Faber-hypercyclic operators. However, some properties of the iterates $T^n = F_n^{\mathbb{D}}(T)$ which are crucial in the proofs of several hypercyclicity statements (like the semigroup property $T^{n+m} = T^n T^m$, for instance) are no longer true in general for $F_n^{\Omega}(T)$. In spite of this, it turns out that most of the results in the spectral theory of hypercyclic operators involving the unit disk/circle have natural analogs for Faber-hypercyclic operators which involve the corresponding open domain Ω or its boundary. An interesting feature in this study is the influence of the geometry of the domain Ω and of the smoothness of its boundary on Ω-hypercyclicity. For another instance of such a relationship between the geometry of the domain and the behaviour of the Faber polynomials of an operator (in relation to the boundary point spectrum $\sigma_p(T) \cap \partial\Omega$), see [2].

1.2. Review of some known results. We recall now some spectral properties of hypercyclic operators involving the unit circle $\mathbb{T}$, and their connections with hypercyclicity. The simplest of these spectral properties appears in the early work of Kitai ([18]):

Theorem 1.2 (Kitai's necessary condition): *If $T \in \mathcal{B}(X)$ is hypercyclic, every connected component of the spectrum $\sigma(T)$ of T meets the unit circle $\mathbb{T}$.*

According to a result of Herrero ([17]), if H is a Hilbert space, the norm-closure in $\mathcal{B}(H)$ of the set $HC(H)$ of hypercyclic operators on H can be completely described in terms of spectral properties of the operator, some of them involving the unit circle. See [17] for the complete statement. One of the main ingredients of Herrero's proof of this result is a criterion for hypercyclicity due to Godefroy and Shapiro ([13]). The Godefroy-Shapiro criterion brought to light the interplay between the behaviour of eigenvectors associated with eigenvalues inside or outside $\mathbb{D}$ and hypercyclicity properties:

Theorem 1.3 (Godefroy-Shapiro criterion): *For any bounded operator T on X, consider the two linear manifolds*

$$H_+(T) = \mathrm{sp}\,[\ker(T - \lambda I) \,:\, |\lambda| > 1]$$

and

$$H_-(T) = \mathrm{sp}\,[\ker(T - \lambda I) \,:\, |\lambda| < 1].$$

If $H_+(T)$ and $H_-(T)$ are dense in X, then T is hypercyclic.

More recently, the connection between properties of the eigenvectors associated to eigenvalues of modulus 1 (such eigenvectors will be called **$\mathbb{T}$-eigenvectors** in the sequel) and properties of the dynamical system (X, T) was studied in [11], [5] and [4]. We quote here one result, which can be seen as the "unimodular counterpart" of the Godefroy–Shapiro Criterion:

Theorem 1.4 ([4]): *Let $T \in \mathcal{B}(X)$ be an operator enjoying the following property: there exists a continuous probability measure σ on the unit circle $\mathbb{T}$ (i.e., such that $\sigma(\{\lambda\}) = 0$ for every $\lambda \in \mathbb{T}$) such that for every subset A of $\mathbb{T}$ with $\sigma(A) = 1$, we have*

$$\overline{\mathrm{sp}}\,[\ker(T - \lambda I) \colon \lambda \in A] = X.$$

Then T is hypercyclic.

When T satisfies the assumption of the above theorem, we say (as in [4]) that T has a perfectly spanning set of $\mathbb{T}$-eigenvectors. A Hilbert space operator with a perfectly spanning set of $\mathbb{T}$-eigenvectors is even **frequently hypercyclic** ([5]): there exists a vector x in X such that for every non-empty open subset U of X, the set of instants $n \in \mathbb{N}$ where $T^n x$ visits U has positive lower density. This notion was studied, for instance, in [5], [6], [8], [7], using in particular some tools of ergodic theory. The corresponding notion of frequent Ω-hypercyclicity is readily defined:

Definition 1.5: A bounded operator T on X is said to be **frequently Ω-hypercyclic** if there exists an $x \in X$ such that for every non-empty open subset U of X,

$$\underline{\text{dens}}\{n \geq 0 : F_n(T)x \in U\} := \liminf_{N \to +\infty} \frac{1}{N} \#\{n \leq N : F_n(T)x \in U\} > 0.$$

1.3. Organization of the paper. Section 2 is mostly expository. We recall the definition of Faber polynomials and some known facts which will be of use in the sequel. We discuss the influence of the geometry of the given domain on the behaviour of the corresponding Faber polynomials inside, outside or on the boundary of the domain. We then present in Section 3 some basic results about Ω-hypercyclicity, for the so-called UB-domains Ω. Under the additional assumption that the boundary of Ω is a rectifiable Jordan curve, we prove the analogue of Kitai's necessary condition: if T is Ω-hypercyclic, every connected component of the spectrum of T meets the boundary of Ω (Proposition 3.3). We then derive a form of the Godefroy–Shapiro Criterion of [13] in this new setting (Theorem 3.4) under the only assumption of the rectifiability of the boundary of the domain. Theorem 3.4 allows us to obtain in Herrero's fashion ([17]) a characterization, in terms of spectral properties, of the norm closure in $\mathcal{B}(H)$ of the set of all Ω-hypercyclic operators on a complex infinite dimensional separable Hilbert space H (Example 3.8). A more surprising fact is that a version of Theorem 1.4 above holds true in the Ω-hypercyclicity setting, and the proof of this fact (for domains with $\mathcal{C}^{1+\alpha}$ boundary) is the object of Section 4 (Theorem 4.2). Section 5 is devoted to the study of frequent Ω-hypercyclicity. We cannot apply directly the ergodic-theoretical methods used for $\mathbb{D}$-hypercyclicity because the new definition does not involve the iterates of a given operator, a crucial point when one is looking forward to applying such tools as Birkhoff's ergodic theorem, for instance. Nonetheless, it turns out that under reasonable assumptions on the **$\partial\Omega$-eigenvectors** of T (eigenvectors of T associated to eigenvalues belonging to the boundary of Ω) and on the smoothness of $\partial\Omega$, the operator T

is frequently Ω-hypercyclic (Theorem 5.1). This is the exact parallel of a statement of [6, Section 5.8]. The proof uses the Frequent Universality Criterion of [8]. This gives examples of operators on Fréchet spaces, such as the translation or differentiation operators on the space of entire functions on $\mathbb{C}$, which are frequently Ω-hypercyclic for every bounded domain Ω with sufficiently smooth boundary (Examples 5.2 and 5.3).

2. Faber polynomials of a domain Ω

We collect in this section some basic facts about Faber polynomials of a domain Ω. Our main reference here is [25], see also [21] or [26].

2.1. Definition and examples. In what follows, Ω will be a bounded domain of the complex plane whose boundary $\partial\Omega = C$ is a closed Jordan curve. Its complement $\overline{\Omega}^c$ being simply connected in the extended complex plane, there exists by the Riemann mapping Theorem a unique function $\psi : \overline{\mathbb{D}}^c \longrightarrow \overline{\Omega}^c$ meromorphic outside $\overline{\mathbb{D}}$ which maps $\overline{\mathbb{D}}^c$ conformally and univalently onto the complement $\overline{\Omega}^c$ of the closure of Ω, and such that $\psi(\infty) = \infty$ and $\psi'(\infty) > 0$. The Laurent expansion of ψ for $|w| > 1$ is of the form

$$\psi(w) = aw + d_0 + d_1/w + d_2/w^2 + \cdots$$

where $a > 0$ is the transfinite diameter or (logarithmic) capacity of Ω. The inverse function ϕ of ψ maps $\overline{\Omega}^c$ conformally and univalently on $\overline{\mathbb{D}}^c$, and ϕ has a Laurent expansion in a neighbourhood of ∞ of the form

$$\phi(z) = (1/a)z + b_0 + b_1/z + b_2/z^2 + \cdots .$$

The n-th **Faber polynomial** F_n^Ω of the domain Ω is the polynomial part of the Laurent expansion of $\phi(z)^n$ at infinity for $n \geq 1$, and F_0^Ω is identically equal to 1. When there is no risk of confusion, we will usually write F_n instead of F_n^Ω. We have

$$\phi(z)^n = F_n(z) + \omega_n(z) \quad \text{for } z \in \overline{\Omega}^c,$$

where ω_n is a bounded analytic function on $\overline{\Omega}^c$ which tends to 0 at infinity. If we denote by C_R the curve $C_R = \{\psi(w) : |w| = R\}$ for $R > 1$, then, for every z in the interior of C_R (in particular, for every z in $\overline{\Omega}$), we have

$$F_n(z) = \frac{1}{2i\pi}\int_{C_R} \frac{\phi(\zeta)^n}{\zeta - z} d\zeta = \frac{1}{2i\pi}\int_{|w|=R} \frac{w^n \psi'(w)}{\psi(w) - z} dw. \tag{1}$$

It follows that for z in the interior of C_R and $|w| > R$,

$$\frac{w\,\psi'(w)}{\psi(w)-z} = 1 + \sum_{n=1}^{+\infty} \frac{F_n(z)}{w^n}, \qquad \text{so that } \frac{\psi'(w)}{\psi(w)-z} = \sum_{n=0}^{+\infty} \frac{F_n(z)}{w^{n+1}}. \tag{2}$$

Therefore the function $w\psi'(w)/(\psi(w)-z)$ is the generating function for the Faber polynomials of Ω. The Faber polynomials of an open disk $\mathbb{D}(z_0, R)$ are given naturally enough by the formula $F_n(z) = ((z-z_0)/R)^n$ for $n \geq 1$. Another case of a domain whose Faber polynomials can be explicitly computed is that of an ellipse with foci at the points -1 and 1 and semi-axes $a = 1/2(R+1/R)$ and $b = 1/2(R-1/R)$ for some $R > 1$. In this case the map ψ is given by the formula

$$\psi(w) = 1/2\,(Rw + 1/(Rw)) \qquad \text{for } |w| > 1,$$

whence it follows that for $n \geq 1$,

$$F_n(z) = \frac{2}{R^n}\,C_n(z),$$

where C_n is the n-th Chebyshev polynomial of the first kind ($C_n(x) = \cos(n \arccos x)$ for $x \in [-1,1]$). We finish this subsection by emphasizing the fact that in general, the Faber polynomials of a given domain Ω are not orthogonal polynomials with respect to a suitable measure on the boundary of Ω, and that composition properties like $z^n \circ z^m = z^{nm}$ are no longer true for Faber polynomials of general domains (see for instance [24] for a characterization of the domains Ω such that $F_n^\Omega \circ F_m^\Omega = F_{nm}^\Omega$ for every n and m).

2.2. Behaviour of $F_n(z)$ when z belongs to Ω. So far, we have imposed almost no restrictions on the smoothness of the curve C. Since C is a Jordan curve, ψ can be extended so as to be continuous on the domain $1 \leq |w| < +\infty$, and then ψ is a homeomorphism between $\mathbb{T}$ and C (this is Caratheodory's theorem, see for instance [22, Ch. 2]). Thus it makes sense to write $\psi(w)$ for $|w| = 1$ and $\phi(z)$ for $z \in C$. If we further suppose that C is rectifiable, $\psi(e^{i\theta})$ is a continuous function of bounded variation, and it admits a derivative $\psi'(e^{i\theta})$ almost everywhere on $\mathbb{T}$ which belongs to L^1 :

$$\int_0^{2\pi} |\psi'(e^{i\theta})| d\theta < +\infty.$$

Then by Lebesgue's dominated convergence theorem, we can make R tend to 1 in (1) for $z \in \Omega$, and we obtain (see for instance [20]) that for every

$n \geq 1$ and $z \in \Omega$

$$(3) \qquad F_n(z) = \frac{1}{2i\pi} \int_{|w|=1} \frac{w^n \psi'(w)}{\psi(w) - z} dw = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i(n+1)\theta} \psi'(e^{i\theta})}{\psi(e^{i\theta}) - z} d\theta.$$

This gives information regarding the asymptotic behaviour of the Faber polynomials inside the domain Ω: for $z \in \Omega$, $F_n(z)$ is the $(n+1)$-th Fourier coefficient of an integrable function on $[0, 2\pi]$, so $F_n(z)$ tends to zero as n goes to infinity. And it is not surprising that the smoother the curve C is, the quicker $F_n(z)$ will tend to zero for $z \in \Omega$.

2.3. Uniform boundedness of $F_n(z)$ on $\overline{\Omega}$. We will need conditions on the geometry of the domain Ω insuring that the polynomials F_n are uniformly bounded on $\overline{\Omega}$. We call such domains UB-domains:

Definition 2.1: The inner domain Ω of a Jordan curve in the complex plane is said to be a **UB-domain** if the Faber polynomials of Ω are uniformly bounded on $\overline{\Omega}$.

Let us give some examples of UB-domains. Every convex domain is a UB-domain. Indeed, if Ω is convex, then ([19]) for every w with $|w| \geq 1$ and every $n \geq 0$, we have

$$|F_n(\psi(w)) - w^n| < 1.$$

In particular, the modulus of F_n is less than 2 on the boundary of Ω, and by the maximum principle, $|F_n(z)| \leq 2$ for every $z \in \overline{\Omega}$.

Other examples of UB-domains are the so-called Faber domains: let $A(\Omega)$ be the Banach algebra of functions which are continuous on $\overline{\Omega}$ and holomorphic in Ω, endowed with the supremum norm. The **Faber operator** T_Ω is the linear operator from $A(\mathbb{D})$ into $A(\Omega)$ mapping each monomial z^n onto the n-th Faber polynomial F_n. The domain Ω is called a **Faber domain** (or a **Faber set**) if T_Ω is a bounded operator. Each Faber domain is a UB-domain, with $|F_n(z)| \leq \|T_\Omega\|$ for every $n \geq 0$ and $z \in \overline{\Omega}$. For instance, piecewise Dini-smooth domains, domains of bounded rotation or domains of bounded secant variation are Faber domains. We refer to [12] (and the references therein) and [22] for all undefined terms. In particular, the inner domains of Jordan curves of class $\mathcal{C}^{1+\alpha}$, $\alpha > 0$, are UB-domains. Recall that, for $p \geq 0$ and $0 < \alpha < 1$, the curve C is of class $\mathcal{C}^{p+\alpha}$ if it has a parametrization $\tau \mapsto z(\tau)$, $0 \leq \tau \leq 2\pi$, which is of class $\mathcal{C}^p$ and such that $z^{(p)}$ is an α-Hölderian function. On the other hand, there are domains with quasiconformal Jordan boundaries which are not UB-domains (see the examples in [19] or [12] of non-Faber domains).

2.4. Behaviour of $F_n(z)$ when z belongs to the complement of $\overline{\Omega}$. We will also need some information on the behaviour of $F_n(z)$ outside the closure of Ω. Suppose that Ω is a bounded domain with simply connected complement. Then ([25, p. 43]) we have $\lim_{n\to\infty} (|F_n(z)|)^{1/n} = |\phi(z)|$ uniformly on every compact subset of $\overline{\Omega}^c$. Since $|\phi(z)| > 1$ on the complement of $\overline{\Omega}$, this implies that $|F_n(z)|$ tends to ∞ for every $z \notin \overline{\Omega}$.

2.5. Behaviour of $F_n(z)$ when z belongs to the boundary of Ω. In order to complete the picture, it remains to study the behaviour of the sequence $(F_n(z))$ when z lies on the boundary of Ω. If $\partial\Omega$ is a Jordan curve, we can write any $z \in C$ as $z = \psi(w)$ for some $w \in \mathbb{T}$, and

$$F_n(z) = F_n(\psi(w)) = w^n - \omega_n(\psi(w)).$$

We are interested in the behaviour of the term $\omega_n(\psi(w))$ as n goes to infinity. If the curve C is sufficiently smooth, $\omega_n(\psi(w))$ goes to zero uniformly in $w \in \mathbb{T}$ as n goes to infinity, and the rate of decay to zero can be explicitly controlled. If the curve C is of class $\mathcal{C}^{p+\alpha}$, the function ψ is of class $\mathcal{C}^p$ and $\psi^{(p)}$ is α-Hölderian on $\mathbb{D}^c$. If $p \geq 1$, ψ' does not vanish on $\mathbb{D}^c$. If C is a curve of class $\mathcal{C}^{p+\alpha}$ with $p \geq 1$ and $\alpha \in]0,1[$, then (see [25, p. 68]) there exists a positive constant M such that for every $n \geq 1$ and $z \in \Omega^c$ ($z \in C = \partial\Omega$, in particular)

$$|\omega_n(z)| \leq M \, \frac{\ln n}{n^{p-1+\alpha}} \cdot \tag{4}$$

Of course if C is an analytic curve, it is easy to show that $\omega_n(z)$ goes to zero exponentially fast on Ω^c: $|\omega_n(z)| \leq M\, r^n$ uniformly in $z \in \Omega^c$ for some $r < 1$.

2.6. Estimating the derivatives of ω_n. In the case when C is an analytic curve, all the derivatives of ω_n also tend to zero exponentially fast on Ω^c (see [23]). If we merely suppose that C is a curve of class $C^{p+\alpha}$ for some $p \geq 1$ and $0 < \alpha < 1$, we can obtain some estimates for the derivatives of ω_n up to the order $p-1$. This fact is mentioned in [23], and seems to belong to the folklore. Since we have been unable to locate in the literature a precise statement or proof, we give in the proposition below the estimate we will need in the sequel.

PROPOSITION 2.2: *Let C be a curve of class $\mathcal{C}^{p+\alpha}$, $p \geq 1$, $0 < \alpha < 1$. Then for every $n \geq 1$, ω_n is a function of class $\mathcal{C}^{p-1}$ on Ω^c (up to the boundary of Ω), and there exists a positive constant M such that for every $k \leq p-1$ and every $z \in \Omega^c$,*

$$|\omega_n^{(k)}(z)| \leq M \frac{\ln n}{n^{p-1-k+\alpha}} \cdot \tag{5}$$

Proof. The proof follows closely the ideas of [25, Ch. 4, p. 64]. For $|w| > 1$, let us write the Laurent expansion for $|t| \geq 1$ of the function

$$t \mapsto h(t,w) = \psi'(t) \cdot \frac{t-w}{\psi(t)-\psi(w)} \quad \text{as} \quad h(t,w) = \sum_{k=0}^{+\infty} \frac{a_k(\psi(w))}{t^k} \cdot$$

Some easy computations show that

$$\omega_n(\psi(w)) = w^n \bigg(\sum_{k=0}^{n} \frac{a_k(\psi(w))}{w^k} - 1 \bigg) \cdot$$

Now

$$\psi'(w)\omega_n'(\psi(w)) = n \frac{\omega_n(\psi(w))}{w} + w^n \sum_{k=0}^{n} \Big(\frac{\psi'(w)a_k'(\psi(w))}{w^k} - \frac{k\, a_k(\psi(w))}{w^{k+1}} \Big) \cdot$$

The first term is easily controlled thanks to estimation (4): for some positive constant M,

$$\left| n \frac{\omega_n(\psi(w))}{w} \right| \leq M \frac{\ln n}{n^{p-2+\alpha}} \quad \text{for every } n \geq 1 \text{ and every } w \in \mathbb{T}.$$

For the second term, consider

$$\frac{\partial}{\partial t} h(t,w) = -\sum_{k=0}^{+\infty} \frac{k\, a_k(\psi(w))}{t^{k+1}}.$$

By [25, Lemma 2, p. 66], the function

$$t \longmapsto \frac{\psi(t)-\psi(w)}{t-w}$$

is of class $\mathcal{C}^{p-1}$ and its $(p-1)$-th derivative with respect to t is α-Hölderian, with a constant in the Hölder condition independent of $w \in \mathbb{T}$. It follows that $t \mapsto h(t,w)$ satisfies the same conditions (recall that ψ' does not vanish on $\mathbb{D}^c$).

Now for $|w| > 1$, the function $h(\,\cdot\,,w)$ has the following Fourier expansion on the unit circle:

$$h(t,w) = \sum_{k=0}^{+\infty} a_k(\psi(w))t^{-k}.$$

The function $\frac{\partial}{\partial t}h(\,\cdot\,,w)$ being of class $\mathcal{C}^{p-2+\alpha}$, the Lebesgue inequality (see for instance [25, Ch. 1, Th. 9]) implies that there exists a constant M (independent of w) such that

$$\left|\frac{\partial}{\partial t}h(t,w) + \sum_{k=0}^{n} \frac{k\,a_k(\psi(w))}{t^{k+1}}\right| \leq M\,\frac{\ln n}{n^{p-2+\alpha}} \qquad \text{for } |w|>1,\ t\in\mathbb{T}.$$

In the same way,

$$\left|\frac{\partial}{\partial w}h(t,w) - \sum_{k=0}^{n} \frac{\psi'(w)a_k'(\psi(w))}{t^{k}}\right| \leq M\,\frac{\ln n}{n^{p-2+\alpha}} \qquad \text{for } |w|>1,\ t\in\mathbb{T}.$$

Now these inequalities are still true for $w, t \in \mathbb{T}$, $w \neq t$, and since $\frac{\partial}{\partial t}h(t,w) + \frac{\partial}{\partial w}h(t,w)$ tends to 0 as t tends to w for every $w \in \mathbb{T}$, we get that

$$\left|\sum_{k=0}^{n}\left(\frac{\psi'(w)a_k'(\psi(w))}{w^k} - \frac{k\,a_k(\psi(w))}{w^{k+1}}\right)\right| \leq M\,\frac{\ln n}{n^{p-2+\alpha}}$$

and

$$|\psi'(w)\,\omega_n'(\psi(w))| \leq M\,\frac{\ln n}{n^{p-2+\alpha}} \qquad \text{for every } n\geq 1 \text{ and every } w\in\mathbb{T}.$$

Since ψ' is bounded away from zero on $\mathbb{T}$, we obtain on $\mathbb{T}$ the estimate for the first derivative of ω_n we were looking for. Since ω_n tends to zero at infinity, the maximum principle yields the same estimate for $|w| \geq 1$. The proof of the corresponding estimates for the other derivatives is done in the same fashion. ∎

3. Some basic properties of Ω-hypercyclic operators

As was already mentioned in the introduction, hypercyclicity implies several spectral restrictions on the operator. We now proceed to investigate the corresponding restrictions entailed by Ω-hypercyclicity. Here is a basic fact to begin with:

FACT 3.1: *Let Ω be a UB-domain, and $T \in \mathcal{B}(X)$ an Ω-hypercyclic operator. Then the point spectrum $\sigma_p(T^*)$ of the adjoint of T is empty.*

Proof. Suppose that $T^*x^* = zx^*$ for some non-zero functional x^* on X and $z \in \mathbb{C}$. Then for every $x \in X$, $\langle x^*, F_n(T)x\rangle = F_n(z)\langle x^*, x\rangle$. Because Ω is a UB-domain, the Faber polynomials are uniformly bounded on $\overline{\Omega}$. Also, $|F_n(z)|$ tends to infinity if z is in the complement of $\overline{\Omega}$. This contradicts the Ω-hypercyclicity of T. ∎

As a straightforward consequence, we obtain

Fact 3.2: *Let Ω be a UB-domain. The operator $T \in \mathcal{B}(X)$ is Ω-hypercyclic if and only if for every non-empty open subsets U and V there exists an integer n such that $F_n(T)(U) \cap V$ is non-empty, and $HC_\Omega(T)$ is then a dense G_δ subset of X.*

Proof. The only thing to prove is that $HC_\Omega(T)$ is dense in X. We use the classical argument of [9]: since $\sigma_p(T^*) = \varnothing$, $p(T)$ has dense range for every non-zero polynomial $p \in \mathbb{C}[\xi]$. Indeed every such non-zero polynomial can be decomposed as a product $p(\xi) = (\xi - z_1)(\xi - z_2)\cdots(\xi - z_r)$ of polynomials of degree 1, and each operator $T - z_i$, $i = 1, \ldots, r$, has dense range. Hence $\{F_n(T)\,p(T)x : n \geq 0\} = p(T)\{F_n(T)x : n \geq 0\}$ is dense in X for every $p \in \mathbb{C}[\xi] \setminus \{0\}$, and $p(T)x$ is an Ω-hypercyclic vector for T. It follows that $HC_\Omega(T)$ is dense in X. ∎

The next step is to derive the analogue of Kitai's necessary spectral condition.

Proposition 3.3: *Let Ω be a UB-domain such that $\partial\Omega$ is a rectifiable Jordan curve, and let $T \in \mathcal{B}(X)$ be an Ω-hypercyclic operator. Then every connected component of the spectrum $\sigma(T)$ of T meets the boundary of Ω.*

Proof. Suppose first that $\sigma(T)$ is connected. If $\sigma(T) \cap \partial\Omega = \varnothing$, then $\sigma(T)$ is either contained in Ω or in $\overline{\Omega}^c$. Suppose that the compact set $\sigma(T)$ is included in Ω, and consider the resolvent function $R(\lambda, T) = (\lambda - T)^{-1}$, which is analytic on $\sigma(T)^c$, in particular on a neighborhood of $\partial\Omega$. If x belongs to X and x^* to X^*, the function $f_{x,x^*}(w) = \langle x^*, R(\psi(w), T)x\rangle$ is well-defined and continuous, hence bounded, on the unit circle. Using the integral representation (3) of the Faber polynomials, we get

$$\langle x^*, F_n(T)x\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n+1)\theta}\psi'(e^{i\theta})\langle x^*, R(\psi(e^{i\theta}), T)x\rangle\, d\theta.$$

Since ψ' in integrable on $[0, 2\pi]$, these are the Fourier coefficients of an L^1 function, so $\langle x^*, F_n(T)x\rangle$ tends to zero as n goes to infinity for every x and x^*, so x cannot be an Ω-hypercyclic vector for T. If the boundary of Ω is an analytic curve, another proof can be given using the methods of [1]: if $\sigma(T) \subset \Omega$, then $\limsup_n \|F_n(T)\|^{1/n} < 1$ so that $\|F_n(T)\|$ tends to zero as n goes to infinity, and T cannot be Ω-hypercyclic. Suppose now that $\sigma(T)$ is contained in the complement of $\overline{\Omega}$. Since ϕ is analytic on this domain, it makes sense to consider the operators $A = \phi(T)$ and $\omega_n(T)$. Also, for every $n \geq 0$, we have

$$F_n(T) - A^n = -\omega_n(T).$$

Moreover since Ω is supposed to be a UB-domain, the Faber polynomials of Ω are uniformly bounded on $\overline{\Omega}$, and hence the functions ω_n are uniformly bounded on the boundary of Ω. Since each function ω_n tends to 0 at infinity, the maximum principle implies that the sequence ω_n is uniformly bounded on the complement of Ω. Using the Dunford–Riesz integral representation of $\omega_n(T)$, it follows, in particular, that for some positive constant C, we have

$$\|F_n(T) - A^n\| \leq C \quad \text{for every } n \geq 0.$$

Hence, if x is an Ω-hypercyclic vector for T, the set $\{A^n x : n \geq 0\}$ is $2C\|x\|$-dense in X in the sense that for every y in X there exists an integer n such that $\|A^n x - y\| \leq 2\,C\,\|x\|$. Now a result of Feldman ([10]) states that whenever an operator has an orbit which is ε-dense on X for some positive ε, this operator is hypercyclic (although the ε-dense orbit itself may not be dense). Feldman's theorem implies that A is hypercyclic, a contradiction since $\sigma(A) = \phi(\sigma(T))$ lies outside the unit disk. This finishes the proof in the case where $\sigma(T)$ is connected. The general case is proved as usual, using the fact that if M is an invariant subspace for T, the operator induced by T on the quotient space X/M is Ω-hypercyclic too. ∎

Our next goal is the following modified Godefroy–Shapiro Criterion, which should by now be quite natural. Here the only condition imposed on the domain is that of rectifiable boundary.

Theorem 3.4: *Let Ω be a bounded domain whose boundary is a rectifiable Jordan curve, and $T \in \mathcal{B}(X)$ a bounded operator on X such that the two following vector spaces*

$$H_+^{\Omega}(T) = \mathrm{sp}\,[\ker(T - zI) : z \in \overline{\Omega}^c\,] \quad \textit{and} \quad H_-^{\Omega}(T) = \mathrm{sp}\,[\ker(T - zI) : z \in \Omega\,]$$

are dense in X. Then T is Ω-hypercyclic.

Proof. The proof follows along the same lines as the classical criterion, using a version of the Universality Criterion ([15]) for the sequence $(F_n(T))_{n\geq 0}$. It suffices to exhibit two dense subsets E and F of X and a sequence $(S_n)_{n\geq 0}$ of maps from F into X (not necessarily linear or continuous) such that the following three conditions are satisfied:

(i) $F_n(T)x$ tends to zero as n goes to infinity for every x in E;
(ii) $S_n y$ tends to zero as n goes to infinity for every y in F;
(iii) $F_n(T)S_n y$ tends to y as n goes to infinity for every y in F.

We choose $E = H^{\Omega}_{-}(T)$ and $F = H^{\Omega}_{+}(T)$, and for $y = \sum_{i=1}^{r} a_i y_i$ with $Ty_i = z_i y_i$, $z_i \in \overline{\Omega}^c$, we set

$$S_n y = \sum_{i=1}^{r} a_i \frac{1}{F_n(z_i)} y_i$$

if every point z_i is not among the zeros of F_n, and $S_n y = 0$ in all other cases. The three conditions above are now easy to check: whenever z is in Ω, $F_n(z)$ goes to zero as n goes to infinity (see Subsection 2.2), and this proves (i). Assertion (ii) follows immediately from the fact that $|F_n(z)|$ tends to infinity when z lies outside $\overline{\Omega}$ (Subsection 2.4). For the same reason, $F_n(z_i)$ is non-zero for all $i = 1, \ldots, r$, if n is large enough, and this implies (iii). ∎

This criterion immediately yields a variety of examples of Ω-hypercyclic operators, with corresponding applications. In all the forthcoming examples, we suppose that Ω is a bounded domain whose boundary is a rectifiable Jordan curve.

Example 3.5: Let $\Phi \in H^\infty(\mathbb{D})$ be a non-constant bounded analytic function on $\mathbb{D}$, and M_Φ the associated multiplier on $H^2(\mathbb{D})$. Then M_Φ^* is Ω-hypercyclic if and only if $\overline{\Phi(\mathbb{D})} \cap \partial\Omega \neq \varnothing$, where $\overline{\Phi(\mathbb{D})}$ denotes the conjugate of the set $\Phi(\mathbb{D})$.

The proof is exactly the same as in [13], using the fact that for every $z \in \mathbb{D}$, $M_\Phi^* k_z = \overline{\Phi(z)} k_z$, where k_z is the reproducing kernel at the point z.

Example 3.6: Let B be the backward shift on ℓ_p, $1 \leq p < +\infty$, or c_0: $Be_0 = 0$ and $Be_n = e_{n-1}$ for $n \geq 1$, where $(e_n)_{n\geq 0}$ denotes the canonical basis of the space. For every complex number ω with $|\omega| > d(0, \partial\Omega)$, ωB is Ω-hypercyclic.

This is an immediate consequence of the computation of the eigenvectors of ωB. As a consequence, we obtain for instance the following:

Example 3.7: Let H be a complex separable infinite dimensional Hilbert space. Every bounded operator on H can be written as the sum of two Ω-hypercyclic operators.

Again, the proof follows along the same lines as in [14], since the proof of [14] uses a decomposition of the form $T = (A + \omega S) + (B - \omega S)$, where S is a direct sum of backward shifts with respect to a suitable orthogonal decomposition of H. And if $|\omega|$ is large enough, Theorem 3.4 applies to $A + \omega S$ and $B - \omega S$, which proves the claim.

We finish this section with a last application of Theorem 3.4, which is a description of the norm closure in $\mathcal{B}(H)$ of the class $HC_\Omega(H)$ of Ω-hypercyclic operators on a complex separable infinite dimensional Hilbert space H:

Example 3.8: The class $\overline{HC_\Omega}(H)$ consists exactly of those operators which satisfy the following three conditions:

(1) $\sigma_W(T) \cap \partial\Omega$ is connected;
(2) $\sigma_0(T) = \varnothing$;
(3) $ind(z - T) \geq 0$ for every $z \in \rho_{SF}(T)$.

Here $\rho_{SF}(T)$ is the semi-Fredholm domain of T,

$$\sigma_W(T) = \sigma(T) \setminus \{\lambda \in \rho_{SF}(T) : ind(\lambda - T) = 0\}$$

is the Weyl spectrum of T, and $\sigma_0(T)$ is the set of normal eigenvalues of T, i.e. isolated eigenvalues such that the corresponding Riesz spectral projection has finite dimensional range. Again, there is almost nothing to change in Herrero's proof ([17]) of this result for $\Omega = \mathbb{D}$. The key point is to show the analogue of Proposition 2.4 of [17]: when

(1) (3) is satisfied,
(2) $\sigma(T)$ and $\sigma_W(T)$ are connected sets,
(3) $T - \alpha$ is a semi-Fredholm operator of positive index for some $\alpha \in \partial\Omega$,

then there exists for every $\varepsilon > 0$ a compact operator K_ε with $\|K_\varepsilon\| < \varepsilon$ such that $T - K_\varepsilon$ is Ω-hypercyclic. Using the notation of [17], we know that K_ε can be chosen so that $T - K_\varepsilon - z$ is semi-Fredholm for every z in a closed disk centered at the point $\alpha \in \partial\Omega$, and that $H_+^\Omega(T - K_\varepsilon)$ and $H_-^\Omega(T - K_\varepsilon)$ are dense in H. Theorem 3.4 then finishes the proof of this point. The rest of the proof is exactly the same as in [17].

4. Faber-hypercyclicity and peripheral point spectrum

We are now concerned with the study of the influence of eigenvectors associated to eigenvalues belonging to the boundary of Ω on the Ω-hypercyclicity of the operator. We call such eigenvectors **$\partial\Omega$-eigenvectors**. The following definition is a mere copy of the definition of a perfectly spanning unimodular eigenvector field given in [4]:

Definition 4.1: Let Ω be a bounded domain of $\mathbb{C}$ whose boundary is a rectifiable Jordan curve, and let $T \in \mathcal{B}(X)$. We say that T has **perfectly spanning $\partial\Omega$-eigenvectors** if there exists a continuous probability measure σ on $\partial\Omega$ such that for every $A \subseteq \partial\Omega$ with $\sigma(A) = 1$, we have

$$\overline{\mathrm{sp}}[\ker(T - z) : z \in A] = X.$$

In view of Theorem 1.4, we would like to show that if $T \in \mathcal{B}(X)$ has perfectly spanning $\partial\Omega$-eigenvectors, then T is Ω-hypercyclic. Contrary to the results of the preceding section, the proofs of which follow a more or less standard pattern, it is not obvious how one should adapt the proof of Theorem 1.4 to the Faber situation. Indeed, the proof of [4] seems to use the unit circle in a crucial way, since everything relies on the behaviour of Fourier coefficients of measures. Surprisingly enough, it turns out that the Ω-version of Theorem 1.4 is still true under some mild smoothness assumptions on the boundary of Ω:

THEOREM 4.2: *Suppose that the boundary of Ω is a curve of regularity $\mathcal{C}^{1+\alpha}$ for some $\alpha \in]0, 1[$. If X is any separable infinite dimensional Banach space, and $T \in \mathcal{B}(X)$ has perfectly spanning $\partial\Omega$-eigenvectors, then T is Ω-hypercyclic.*

Proof. The proof starts along the same lines as in [4]. Let σ be a continuous measure on $\partial\Omega$ satisfying the assumptions of Theorem 4.2. Just as in Lemma 2.7 of [4], it follows from the Kuratowski–Ryll–Nardzewski Theorem that there exists a countable family $(E_i)_{i\geq 1}$ of eigenvector fields $E_i : \partial\Omega \to X$ which are σ-measurable such that $\sup_{z\in\partial\Omega} \|E_i(z)\| \leq 1$ and for every $z \in \partial\Omega$,

$$\ker(T - z) = \overline{\mathrm{sp}}\,[E_i(z) : i \geq 1].$$

Let now $(f_l)_{l\geq 1}$ be a dense sequence of smooth (C^∞, for instance) functions in $L^2(\partial\Omega,\sigma)$, and consider for $i\geq 1$, $l\geq 1$,

$$x_l^{(i)}=\int_{\partial\Omega} f_l(\zeta)\,E_i(\zeta)\,d\sigma(\zeta).$$

We have for every $n\geq 0$

$$F_n(T)x_l^{(i)}=\int_{\partial\Omega} F_n(\zeta)\,f_l(\zeta)\,E_i(\zeta)\,d\sigma(\zeta).$$

Since $\partial\Omega$ is of class $\mathcal{C}^{1+\alpha}$, there exists (see Section 2.5) a positive constant M such that for every $n\geq 1$ and every $z\in\partial\Omega$,

$$|\omega_n(z)|\leq M\,(\ln(n)/n^\alpha)\cdot$$

In particular, ω_n tends to 0 uniformly on $\partial\Omega$. This implies the existence of an integer n_0 such that $|F_n(\psi(w))-w^n|<1$ for every $n\geq n_0$ and every $w\in\mathbb{T}$, which makes it possible to define

$$S_n x_l^{(i)}=\int_{\partial\Omega}\frac{1}{F_n(\zeta)}\,f_l(\zeta)\,E_i(\zeta)\,d\sigma(\zeta)$$

for $n\geq n_0$, while for $n<n_0$ we set $S_nx_l^{(i)}=0$. Clearly $F_n(T)S_nx_l^{(i)}=x_l^{(i)}$ for $n\geq n_0$. The vectors $x_l^{(i)}$, $i,l\geq 1$, span a dense subspace of X. Indeed, if $\langle x^*,x_l^{(i)}\rangle=0$ for every $i,l\geq 1$, then $\langle x^*,E_i(\,\cdot\,)\rangle=0$ σ-almost surely for every $i\geq 1$, and since the $\partial\Omega$-eigenvectors are σ-spanning, we get $x^*=0$. Now it suffices to exhibit a sequence (n_k) such that $F_{n_k}(T)x_l^{(i)}$ and $S_{n_k}x_l^{(i)}$ tend to zero as n_k tends to infinity for every $l,i\geq 1$, and the Universality Criterion will then conclude the proof. Recall that $\psi:\overline{\mathbb{D}}^c\longrightarrow\overline{\Omega}^c$ extends to a continuous function on the unit circle which maps $\mathbb{T}$ univalently onto $\partial\Omega$ (it is even of class $C^{1+\alpha}$ under our smoothness assumption). Let μ be the measure on $\mathbb{T}$ defined by $\mu(B)=\sigma(\psi(B))$ for any measurable subset B of $\mathbb{T}$: it is a probability measure on $\mathbb{T}$, which is continuous, and we have

$$F_n(T)x_l^{(i)}=\int_{\mathbb{T}} F_n(\psi(\lambda))\,f_l(\psi(\lambda))\,E_i(\psi(\lambda))d\mu(\lambda).$$

Now for every $\lambda\in\mathbb{T}$ and every $n\geq 0$, we have

$$\lambda^n=F_n(\psi(\lambda))+\omega_n(\psi(\lambda)).$$

So our formula for $F_n(T)x_l^{(i)}$ becomes

$$\begin{aligned}(6)\qquad F_n(T)x_l^{(i)} &= \int_{\mathbb{T}} \lambda^n\, f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\mu(\lambda) \\ &\quad - \int_{\mathbb{T}} \omega_n(\psi(\lambda))\, f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\mu(\lambda).\end{aligned}$$

We decompose in the same fashion the expression of $S_n x_l^{(i)}$ for $n \geq n_0$:

$$S_n x_l^{(i)} = \int_{\mathbb{T}} \frac{1}{F_n(\psi(\lambda))}\, f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\mu(\lambda).$$

We have

$$\frac{1}{F_n(\psi(\lambda))} = \frac{1}{\lambda^n} + \frac{\omega_n(\psi(\lambda))}{F_n(\psi(\lambda))\, \lambda^n},$$

so that

$$\begin{aligned}(7)\qquad S_n x_l^{(i)} &= \int_{\mathbb{T}} \lambda^{-n}\, f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\mu(\lambda) \\ &\quad + \int_{\mathbb{T}} \frac{\omega_n(\psi(\lambda))}{F_n(\psi(\lambda))\, \lambda^n}\, f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\mu(\lambda).\end{aligned}$$

The second term in each one of the expressions (6) and (7) goes to zero as n goes to infinity: indeed, ω_n tends to zero uniformly on $\partial\Omega$ by our smoothness assumption on $\mathcal{C}$, which proves the statement for (6), and for (7) it suffices to show that the quantity

$$\frac{\omega_n(\psi(\lambda))}{F_n(\psi(\lambda))\, \lambda^n}$$

tends to zero uniformly on the boundary of Ω, for instance. But since

$$\left|\frac{\omega_n(\psi(\lambda))}{F_n(\psi(\lambda))}\right| = \left|\frac{\omega_n(\psi(\lambda))}{\lambda^n - \omega_n(\psi(\lambda))}\right| \leq \frac{||\omega_n||_{\infty,\partial\Omega}}{1 - ||\omega_n||_{\infty,\partial\Omega}} \leq M\, \frac{\ln n}{n^\alpha},$$

the uniform convergence to zero on $\partial\Omega$ follows. Thus the study of $F_n(T)x_l^{(i)}$ (respectively $S_n x_l^{(i)}$) boils down to the study of the $(-n)$-th (respectively n-th) Fourier coefficient of the vector-valued function $\lambda \mapsto f_l(\psi(\lambda))\, E_i(\psi(\lambda))$ with respect to the measure μ. Proceeding just as in [4], we obtain the existence of a sequence (n_k) of integers such that

$$\int_{\mathbb{T}} \lambda^{n_k}\, f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\mu(\lambda) \quad \text{and} \quad \int_{\mathbb{T}} \lambda^{-n_k}\, f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\mu(\lambda)$$

tend to 0 as k goes to infinity for every $i, l \geq 1$. The main ingredient of the proof is Wiener's Theorem, which can be applied because the measure μ is continuous. We refer the reader to [4] for details. This completes the proof of Theorem 4.2. ∎

5. Frequent Ω-hypercyclicity

This section is devoted to the study of frequent Ω-hypercyclicity. The main idea of Theorem 5.1 below comes from a frequent hypercyclicity result of [6], which runs as follows: if $T \in \mathcal{B}(X)$ is an operator whose unimodular eigenvectors can be described through a countable family $(E_i)_{i\geq 1}$ of functions $E_i : \mathbb{T} \longrightarrow X$ of class C^2 such that for every $\lambda \in \mathbb{T}$,

$$\ker(T-\lambda) = \overline{\text{sp}}\,[E_i(\lambda) \,:\, i \geq 1]$$

and $\overline{\text{sp}}\,[\ker(T-\lambda) \,:\, \lambda \in \mathbb{T}] = X$, then T is frequently hypercyclic. This is a direct consequence (see [6]) of the Frequent Hypercyclicity Criterion of [5] or [8], and since many operators have C^2 unimodular eigenvector fields, this result can be applied in various situations, even in Fréchet spaces (see for instance [7]). Recall that an F-space is a Fréchet space whose topology is defined thanks to a translation invariant metric (F-norm) $\|\cdot\|$. Our aim is to prove the following theorem for frequent Ω-hypercyclicity:

Theorem 5.1: *Let Ω be a bounded domain of $\mathbb{C}$ whose boundary is a curve of class $C^{3+\alpha}$ for some $\alpha \in]0,1[$, and T a continuous linear operator on a separable infinite dimensional F-space X. Suppose that there exists a countable family of functions $(E_i)_{i\geq 1}$ defined on $\partial\Omega$ with values in X, of class C^2, such that for every $z \in \partial\Omega$ we have $\ker(T-z) = \overline{\text{sp}}\,[E_i(z) \,:\, i \geq 1]$. We also suppose that the $\partial\Omega$-eigenvectors are spanning, i.e.*

$$X = \overline{\text{sp}}\,[\ker(T-z) \,:\, z \in \partial\Omega].$$

Then T is frequently Ω-hypercyclic.

The proof relies on the Frequent Universality Criterion of [8], which we state here in the context of Faber hypercyclicity:

Frequent Faber-hypercyclicity Criterion: Let Ω be a bounded domain of $\mathbb{C}$ whose boundary is a closed Jordan curve, and let $(F_n)_{n\geq 0}$ the sequence of Faber polynomials of Ω. Let X be a separable F-space and T a continuous operator on X. Suppose that there exist a dense sequence $(x_l)_{l\geq 1}$ of vectors of

X and a sequence $(S_n)_{n\geq 1}$ of maps defined on the set $\{x_l\ ;\ l\geq 1\}$ such that the following three conditions are fulfilled:

(1) for every $l \geq 1$, the series $\sum_{n\geq 0} F_{n+k}(T)S_k x_l$ converges unconditionally uniformly in $k \geq 0$;
(2) for every $l \geq 1$, the series $\sum_{n\geq 0} S_{n+k}F_k(T)x_l$ converges unconditionally uniformly in $k \geq 0$;
(3) for every $l \geq 1$, $F_n(T)S_n x_l$ tends to x_l as n tends to infinity.

Then T is frequently Ω-hypercyclic.

Note that as soon as condition (2) above is satisfied, the series $\sum_{n\geq 0} S_n x_l$ converges unconditionally in X for every $l \geq 1$. Saying, for instance, that the series $\sum_{n\geq 0} F_{n+k}(T)S_k x_l$ converges unconditionally uniformly in $k \geq 0$ means that for every $\varepsilon > 0$, there exists an integer N_0 such that for every finite subset F of $\mathbb{N}$ with $F \cap [0, N_0] = \varnothing$ and every $k \geq 0$, we have

$$\left\| \sum_{n\in F} F_{n+k}(T)S_k x_l \right\| < \varepsilon.$$

Proof of Theorem 5.1. Let $(f_l)_{l\geq 1}$ be a dense sequence of C^2 functions in $L^2(\partial\Omega, \mu)$ where $\mu = \psi(d\lambda)$, $d\lambda$ being the normalized length measure on the unit circle. We set

$$x_l^{(i)} = \int_{\partial\Omega} f_l(\zeta)\, E_i(\zeta)\, d\mu(\zeta) = \int_{\mathbb{T}} f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\lambda.$$

We have seen that the vectors $x_l^{(i)}$, $i, l \geq 1$, span a dense subspace of X, and thus it suffices to check the assumptions of the Frequent Faber-hypercyclicity Criterion on these vectors $x_l^{(i)}$. We have for every $n \geq 0$

$$F_n(T)x_l^{(i)} = \int_{\mathbb{T}} F_n(\psi(\lambda))\, f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\lambda.$$

The smoothness condition on the boundary implies that ω_n tends to zero uniformly on $\partial\Omega$, and therefore there exists $n_0 \in \mathbb{N}$ such that $|F_n(\psi(\lambda)) - \lambda^n| < 1$ for every $n \geq n_0$ and every $\lambda \in \mathbb{T}$. In particular, $F_n(\psi(\lambda))$ does not vanish on $\mathbb{T}$ for every $n \geq n_0$. Just as in the proof of Theorem 4.2 above, we set

$$S_n x_l^{(i)} = \int_{\mathbb{T}} \frac{1}{F_n(\psi(\lambda))}\, f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\lambda \quad \text{ for } n \geq n_0$$

and $S_n x_l^{(i)} = 0$ for $n < n_0$. Then condition (3) of the criterion above is clearly satisfied. In all the arguments below we will assume that $n \geq n_0$. We have to study the quantities

$$F_{n+k}(T)S_k x_l^{(i)} = \int_{\mathbb{T}} \frac{F_{n+k}(\psi(\lambda))}{F_k(\psi(\lambda))} f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\lambda.$$

Since $\|\omega_k\|_{\infty,\partial\Omega}$ goes to zero as k goes to infinity, there exist $k_0 \geq 0$ and $\delta > 0$ such that $|\lambda^k - \omega_k(\psi(\lambda))| \geq \delta$ for every $\lambda \in \mathbb{T}$ and $k \geq k_0$. We have for $\lambda \in \mathbb{T}$ and $n \geq n_0$, $k \geq k_0$,

$$\frac{F_{n+k}(\psi(\lambda))}{F_k(\psi(\lambda))} = \lambda^n - \frac{\omega_{n+k}(\psi(\lambda))}{\lambda^k - \omega_k(\psi(\lambda))} + \lambda^n \frac{\omega_k(\psi(\lambda))}{\lambda^k - \omega_k(\psi(\lambda))},$$

so that the expression for $F_{n+k}(T)S_k x_l^{(i)}$ can be rewritten as

$$\begin{aligned} F_{n+k}(T)S_k x_l^{(i)} = &\int_{\mathbb{T}} \lambda^n f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\lambda \\ &- \int_{\mathbb{T}} \frac{\omega_{n+k}(\psi(\lambda))}{\lambda^k - \omega_k(\psi(\lambda))} f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\lambda \\ &+ \int_{\mathbb{T}} \lambda^n f_l(\psi(\lambda))\, E_{i,k}(\psi(\lambda))(\psi(\lambda))\, d\lambda, \end{aligned} \tag{8}$$

where

$$E_{i,k}(\psi(\lambda)) = \frac{\omega_k(\psi(\lambda))}{\lambda^k - \omega_k(\psi(\lambda))} E_i(\psi(\lambda)).$$

The first term in (8) is easy to control: $f_l \circ \psi \cdot E_i \circ \psi$ is a function of class C^2. This implies the existence of a positive constant C_1 such that for every $n \geq n_0$,

$$\left\| \int_{\mathbb{T}} \lambda^n f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\lambda \right\| \leq C_1/n^2 \cdot$$

The second term is also small uniformly in $k \geq k_0$, due to the fact that the boundary of Ω is of class $C^{3+\alpha}$. Indeed, we have

$$\|\omega_{n+k}\|_{\infty,\partial\Omega} = O\left(\ln(n+k)/(n+k)^{2+\alpha}\right)$$

and $|\lambda^k - \omega_k(\psi(\lambda))| \geq \delta$ for every $\lambda \in \mathbb{T}$ and $k \geq k_0$. This yields the existence of a constant $C_2 > 0$ such that for every $k \geq k_0$ and $n \geq 1$,

$$\left\| \int_{\mathbb{T}} \frac{\omega_{n+k}(\psi(\lambda))}{\lambda^k - \omega_k(\psi(\lambda))} f_l(\psi(\lambda))\, E_i(\psi(\lambda))\, d\lambda \right\| \leq C_2/n^2 \cdot$$

The remaining difficulty is to estimate the third term in (8) uniformly in k. There exists a constant $M > 0$ such that for every $n \geq n_0$ and $k \geq k_0$,

$$\left\| \int_{\mathbb{T}} \lambda^n f_l(\psi(\lambda))\, E_{i,k}(\psi(\lambda))\, d\lambda \right\| \leq \frac{M}{n^2} \|(E_{i,k} \circ \psi)''\|_{\infty,\mathbb{T}},$$

so the issue is to bound the second-order derivatives of $E_{i,k} \circ \psi$ independently of k. Since the boundary of Ω is of class $C^{3+\alpha}$, ω_k is twice differentiable up to the boundary of Ω, and by Proposition 2.2 we have

$$\|\omega_k''\|_{\infty,\partial\Omega} = O\left(\ln k / k^{\alpha}\right).$$

Then it is not difficult to see that there exists a positive constant C_3 such that for every large n and $k \geq k_0$,

$$\left\| \int_{\mathbb{T}} \lambda^n \, f_l(\psi(\lambda)) \, E_{i,k}(\psi(\lambda)) \, d\lambda \right\| \leq C_3/n^2.$$

It follows that the series $\sum_{n \geq 0} \|F_{n+k}(T) S_k x_l^{(i)}\|$ converges uniformly in $k \geq k_0$. Since this series is also convergent for every $k < k_0$, the convergence is uniform in $k \geq 0$, and assumption (1) of the Frequent Faber-hypercyclicity Criterion is satisfied. Assumption (2) is proved in the same fashion. ∎

Again, this gives many examples of frequently Ω-hypercyclic operators: the operators of Examples 3.5, 3.6 and 3.7, for instance. Other examples can be constructed from the operators of [5], [8] or [7]. We leave the reader to write these down by himself, and we concentrate below on two examples of operators on Fréchet spaces which are frequently Ω-hypercyclic for *every* simply connected bounded domain Ω with $\mathcal{C}^{3+\alpha}$ boundary. Clearly such operators cannot exist on Banach spaces by Proposition 3.3.

Example 5.2: The translation operator $T : f \longmapsto [z \mapsto f(z+1)]$ on the space $\mathcal{O}(\mathbb{C})$ of entire functions on $\mathbb{C}$ is frequently Ω-hypercyclic for every simply connected bounded domain Ω with $\mathcal{C}^{3+\alpha}$ boundary.

Proof. For every $\lambda \in \mathbb{C}$, let f_λ be the function defined by $f_\lambda(z) = e^{\lambda z}$. We have $T f_\lambda = e^{\lambda} f_\lambda$. If Ω satisfies the assumptions above, let $\tau \mapsto z(\tau)$, $0 \leq \tau \leq 1$, be a parametrization of $\partial\Omega$ of class $\mathcal{C}^{3+\alpha}$. If 0 belongs to $\partial\Omega$, we choose $z(0) = z(1) = 0$. Let $\gamma = \{z(\tau), 1/4 \leq \tau \leq 3/4\}$. It makes sense to consider for $1/4 \leq \tau \leq 3/4$ the function $F(\tau) = r(\tau) f_{\log z(\tau)}$, where log is a determination of the logarithm along the arc γ, and r is a smooth bump function on $[0,1]$ whose support is the segment $[1/4, 3/4]$ and which is positive on $]1/4, 3/4[$. Then $TF(\tau) = z(\tau)F(\tau)$, and we obtain in this way an eigenvector field for T on $\partial\Omega$ of class $\mathcal{C}^{3+\alpha}$. In order to apply Theorem 5.1, it suffices to prove that the linear span of the functions $F(\tau)$ is dense in $\mathcal{O}(\mathbb{C})$. This follows for instance from an argument of [7]. ∎

Example 5.3: The differentiation operator $D : f \mapsto f'$ on $\mathcal{O}(\mathbb{C})$ is frequently Ω–hypercyclic for every simply connected bounded domain Ω with $\mathcal{C}^{3+\alpha}$ boundary.

The argument is the same. These two examples show another instance of the fact that the translation and differentiation operators on $\mathcal{O}(\mathbb{C})$ are "more hypercyclic" that most operators on Banach spaces (see for instance [8], [3]). We do not know whether there exists a function $f \in \mathcal{O}(\mathbb{C})$ which is Ω–hypercyclic (or even frequently Ω–hypercyclic) for T (or D) for *every* simply connected bounded domain Ω with smooth boundary.

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